Lecture I: Exact BS quantization condition

S. Lukyanov

Introduction: IQFT/IPDE correspondence

- Correspondence principle: when Planck constant $\hbar \to 0$

  Quantum theory $\longrightarrow$ Classical theory

- Since 1998, a new type of (mathematical) correspondence

  Integrable Quantum Field Theory (IQFT) $\longleftrightarrow$ Integrable Partial Differential Equations (IPDE)

- Historically, the first example is the so-called ODE/IM correspondence

- Many faces of the IQFT/IPDE correspondence. In particular, the appearance of the Painlevé transcendents in the description of the monodromy of linear ODEs, say

  -The correspondence between a quantum mechanical particle in a cosine potential and Painlevé III

  -The correspondence between a quasiclassical conformal block and Painlevé VI

3D harmonic oscillator

Any story in physics should began with the harmonic oscillator, a problem which every physicist know. So, let me start with the three dimensional (3D) harmonic oscillator:

$$\left[-\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}^2 + U(\boldsymbol{r}) \right] \Psi = \mathcal{E}\,\Psi\ , \qquad U(\boldsymbol{r}) = \frac{m\omega^2}{2}\,\boldsymbol{r}^2$$

- $U(\boldsymbol{r}) = \frac{m\omega^2}{2}\,(x_1^2 + x_2^2 + x_3^2)\ :\quad \mathcal{E}_{\boldsymbol{n}} = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right)$
- $U = U(|\boldsymbol{r}|) \Longrightarrow$ separation of variables:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \underbrace{\frac{\ell(\ell+1)}{z^2}}_{\text{centrifugal potential}} + z^2 \right] \psi = E\,\psi \qquad \left(z = \sqrt{\frac{m\omega}{\hbar}}\,|\boldsymbol{r}|,\ E = \frac{2}{\hbar\omega}\,\mathcal{E} \right)$$

$$E_n = 2\,\underbrace{(2n+\ell)}_{n_1+n_2+n_3} + 3 \qquad (n = 0, 1, 2, \ldots)$$

Here ℓ is the orbital momentum.

3D anharmonic oscillator

Let us make the problem slightly less trivial and consider the "anharmonic" oscillator:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+U_{\rm eff}(z)\right]\psi=E\,\psi\ ,\qquad U_{\rm eff}(z)=\frac{\ell(\ell+1)}{z^2}+z^{2\alpha}\qquad(\alpha>0)$$

Particular cases (see ref. [1])

- Harmonic oscillator: $\alpha=1$ (Hermite)
- Infinite Spherical Potential Well: $\alpha=\infty$ (Bessel)

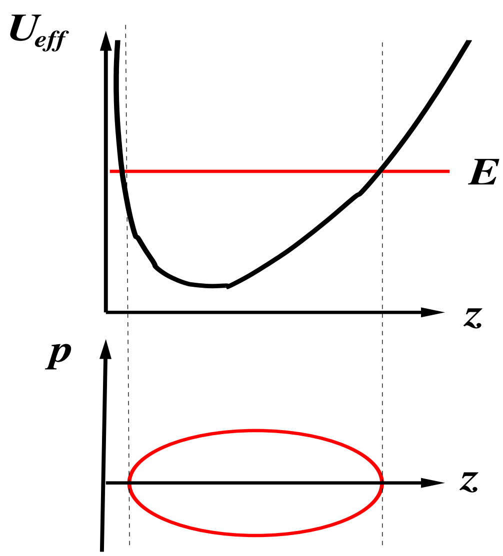
$$z^{2\alpha}|_{\alpha\to+\infty}\to\begin{cases}0\ ,\ \ 0\leq z<1\\ \infty\ ,\quad z>1\end{cases}$$

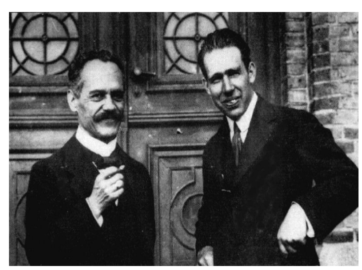
- $\ell=0$, $\alpha=\frac{1}{2}$ (Airy):

$$U_{\rm eff}(z)=z\qquad\text{(constant 1D force)}$$

Generally speaking, this is a confining potential with the discrete spectrum.

Bohr-Sommerfeld (BS) quantization condition





$$\oint\frac{\mathrm{d}z}{2\pi}\,p(z)=n+\frac{1}{2}$$

Langer's correction (1937)

Exercise: Using the BS quantization condition show that

$$E_n\approx C_0\,\left(n+\tfrac{1}{4}\,(2\ell+3)\right)^{\frac{2\alpha}{\alpha+1}}\qquad(n\gg1)$$

and find the n and ℓ-independent constant $C_0=C_0(\alpha)$ explicitly. Show that for $\alpha=1$ the BS quantization condition turns out to be exact.

Remarkably, the problem is "integrable" in a certain sense. Namely we can find the "exact" Bohr-Sommerfeld quantization condition. It requires the notion of the **"Spectral Determinant"**.

Spectral determinant

- Characteristic polynomial for a linear operator (matrix):

$$\mathrm{Det}(\hat{A} - \lambda\,\hat{I}) = \mathrm{Det}(\hat{A})\,\prod_n (1 - \lambda/\lambda_n)$$

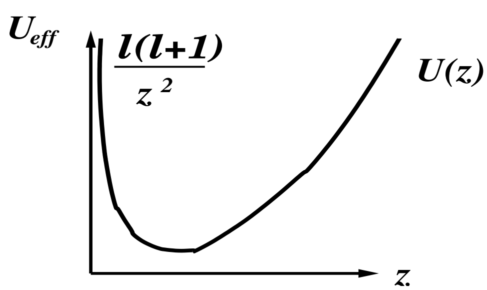
- $\hat{H} = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\ell(\ell+1)}{z^2} + z^{2\alpha}$:

$$D(E) \equiv \mathrm{Det}(\hat{H} - E\,\hat{I}) = \underbrace{D(0)}_{\mathrm{Det}(\hat{H})}\,\prod_{n=0}^{\infty}(1 - E/E_n)$$

- $E_n \sim n^{\frac{2\alpha}{\alpha+1}}\quad (n \to \infty) \Longrightarrow$ the product converges for $\alpha > 1$

Calculation of the spectral determinant

Two solutions of the Shrödinger equation for the confining potential



$$U_{\mathrm{eff}}(z) = \frac{\ell(\ell+1)}{z^2} + \underbrace{z^{2\alpha}}_{U(z)}$$

$$\psi(z) \to z^{\ell+1} \qquad \text{as} \qquad z \to 0$$

$$\chi(z) \approx \frac{1}{\sqrt{U(z)}}\exp\left(-\int^z \mathrm{d}z\sqrt{U(z)}\right) = z^{-\alpha/2}\ \exp\left(-\frac{z^{1+\alpha}}{1+\alpha}\right) \qquad (z \to +\infty)$$

(WKB asymptotic)

Properties of the Wronskian $W[\chi, \psi] \equiv \chi\psi' - \chi'\psi$

- W does not depend on z, i.e., $W = W(E, \ell)$
- $W(E, \ell)$ is an entire function of E (analytic in the whole complex plane)
- $W(E_n, \ell) = 0 \quad (\chi \propto \psi_+$ as $E = E_n)$

Exercise: Show that the spectral determinant $\mathbf{D}(E)$ can be identified with the Wronskian $W(E, \ell)$, i.e.,

$$\boldsymbol{W(E,\ell) = const\ \mathrm{D}(E) \equiv \mathrm{Det}(\hat{H} - E\,\hat{I})}$$

Elements of Regge theory for $U_{\text{eff}}(z|\ell) = \frac{\ell(\ell+1)}{z^2} + z^{2\alpha}$

In quantum mechanics, Regge theory is the study of the analytic properties of scattering amplitudes as functions of angular momentum, where the angular momentum is not restricted to be an integer but is allowed to take any complex value. The nonrelativistic theory was developed by Tullio Regge in 1959 (see § 141 in ref. [1]).

IL NUOVO CIMENTO VOL. XIV, N. 5 1° Dicembre 1959

Introduction to Complex Orbital Momenta.

T. REGGE

Max-Planck-Institut für Physik und Astrophysik - München (*)

(ricevuto il 18 Luglio 1959)

Summary. — In this paper the orbital momentum j, until now considered as an integer discrete parameter in the radial Schrödinger wave equations, is allowed to take complex values. The purpose of such an enlargement is not purely academic but opens new possibilities in discussing the connection between potentials and scattering amplitudes. In particular it is shown that under reasonable assumptions, fulfilled by most field theoretical potentials, the scattering amplitude at some fixed energy determines the potential uniquely, when it exists. Moreover for special classes of potentials $V(x)$, which are analytically continuable into a function $V(z)$, $z = x + iy$, regular and suitable bounded in $x > 0$, the scattering amplitude has the remarcable property of being continuable for arbitrary negative and large cosine of the scattering angle and therefore for arbitrary large real and positive transmitted momentum. The range of validity of the dispersion relations is therefore much enlarged.

Figure 1: Tullio Regge 1931-2014

Here are some facts from Regge theory for $U_{\text{eff}}(z|\ell) = \frac{\ell(\ell+1)}{z^2} + z^{2\alpha}$

- $\chi(z|\ell) \to 0 \quad (z \to \infty): \quad \chi(z|\ell) = \chi(z|-1-\ell)$

  is an entire function of the complex variable ℓ

- $\psi(z|\ell) \to z^{\ell+1} \quad (z \to 0)$ is a meromorphic function of ℓ.

  Only simple (Regge) poles are allowed for $\Re e(\ell) < -\frac{1}{2}$

- $\psi_+(z) \equiv \psi(z|\ell)$, $\psi_-(z) = \psi(z|-\ell-1)$ – two linear independent solutions: $(\psi_-\psi'_+ - \psi_+\psi'_-)|_{z\to 0} = 2\ell + 1$

- $$\boldsymbol{\chi(z) = \frac{1}{2\ell+1}\big(W_+\,\psi_-(z) - W_-\,\psi_+(z)\big), \quad W_\pm = W[\chi, \psi_\pm]}$$

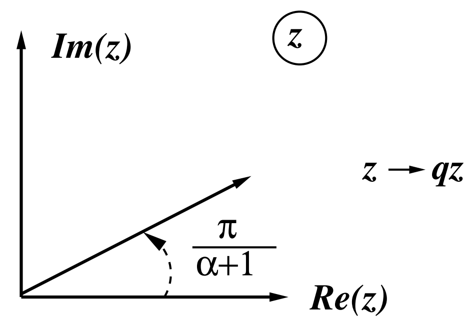
- $W_-(E,\ell) = W_+(E,-\ell-1)$

Remarkable symmetry of the anharmonic oscillator

Let $z \mapsto q\,z$, then

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\ell(\ell+1)}{z^2} + z^{2\alpha} - E \mapsto \frac{1}{q^2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\ell(\ell+1)}{z^2} + \boldsymbol{q^{2\alpha+2}}\, z^{2\alpha} - q^2 E\right]$$

If $\boldsymbol{q}^{2\alpha+2} \mathbf{= 1}$ i.e. $\boldsymbol{q} = \mathbf{e}^{\frac{i\pi}{\alpha+1}}$:



$$\hat{\Omega}: \quad \boldsymbol{z} \mapsto \boldsymbol{q}\,\boldsymbol{z}\ , \quad \boldsymbol{E} \mapsto \boldsymbol{q}^{-2}\,\boldsymbol{E} \quad \left(\boldsymbol{q} = \mathrm{e}^{\frac{\mathrm{i}\pi}{\alpha+1}}\right) \ \text{is a symmetry}$$

Derivation of the "Quantum Wronskian" relation [2]

- $\psi_+(z) \to z^{\ell+1}$: $\quad \hat{\Omega}\psi_+(z) = q^{\ell+1}\ \psi_+(z)\ , \quad \hat{\Omega}\psi_-(z) = q^{-\ell}\ \psi_-(z)$

- $\chi(z) = \frac{1}{2\ell+1}\Big[W_+(E)\ \psi_-(z) - W_-(E)\ \psi_+(z)\Big]$:

  $\hat{\Omega}\chi(z) = \frac{1}{2\ell+1}\Big[W_+(q^{-2}E)\ q^{-\ell}\psi_-(z) - W_-(q^{-2}E)\ q^{\ell+1}\psi_+(z)\Big]$

- $W[\chi, \hat{\Omega}\chi] = \frac{1}{2\ell+1}\Big[q^{-\ell}W_-(E)W_+(q^{-2}E) - q^{\ell+1}W_+(E)W_-(q^{-2}E)\Big]$

- $\chi(z) \to z^{-\alpha/2}\ \exp\big(-\frac{z^{1+\alpha}}{1+\alpha}\big)$: $\quad W[\chi, \hat{\Omega}\chi]|_{z\to+\infty} = 2$

$$\boldsymbol{q^{-\ell}\, W_-(E)W_+(q^{-2}E) - q^{\ell+1}\, W_+(E)W_-(q^{-2}E) = 2(2\ell+1)}$$

Derivation of the exact BS quantization condition

- $q^{-\ell}\, W_-(E)W_+(q^{-2}E) - q^{\ell+1}\, W_+(E)W_-(q^{-2}E) = 2(2\ell+1)$

- $q^{-\ell}W_-(q^2E)W_+(E) - q^{\ell+1}W_+(q^2E)W_-(E) = 2(2\ell+1) \quad (E \mapsto q^2E)$

- $$W_+(E_n) = 0 \quad : \quad \begin{array}{c} q^{-\ell}W_-(E_n)W_+(q^{-2}E_n) = 2(2\ell+1) \\ \\ -q^{\ell+1}W_+(q^2E_n)W_-(E_n) = 2(2\ell+1) \end{array}$$

- $$\frac{W_+(q^{-2}E_n)}{W_+(q^2E_n)} = -q^{2\ell+1} \qquad (q = \mathrm{e}^{\frac{\mathrm{i}\pi}{\alpha+1}})$$

- $W_+(E) = D(E) \equiv D(0)\, \prod_{n=0}^{\infty} \big(1 - E/E_n\big)$

$$\frac{D(q^{-2}E_n)}{D(q^2E_n)} = -q^{2\ell+1} \qquad \big(\text{does not depend on } D(0)!\big)$$

- $Q(E) = E^{\frac{1}{4}(2\ell+1)}\; D(E)$:

$$\frac{Q(q^{-2}E_n)}{Q(q^2E_n)} = -1 \quad \text{i.e.} \quad \frac{1}{2\pi\mathrm{i}}\;\log\left(\frac{Q(q^{-2}E_n)}{Q(q^2E_n)}\right) = N_n + \frac{1}{2}$$

where N_n are some integers.

- For $\alpha = 1$: $\qquad N_n = n$

$$\boxed{\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}\mathbf{i}}\;\mathbf{log}\left(\frac{\boldsymbol{Q(q^{-2}E_n)}}{\boldsymbol{Q(q^2E_n)}}\right) = \boldsymbol{n} + \frac{\mathbf{1}}{\mathbf{2}} \qquad \left(\boldsymbol{q} = \mathbf{e}^{\frac{\mathbf{i}\boldsymbol{\pi}}{\boldsymbol{\alpha}+\mathbf{1}}},\ \mathbf{n = 0, 1, 2\ldots}\right)}$$

- In the WKB approximation

$$n + \frac{1}{2} = \frac{1}{2\pi\mathrm{i}}\;\log\left(\frac{Q(q^{-2}E_n)}{Q(q^2E_n)}\right) \approx \oint \frac{\mathrm{d}z}{2\pi}\; p(z)$$

- It allows one to develop a systematic large-n expansion:

$$E_n \asymp (4n+2\ell+3)^{\frac{2\alpha}{\alpha+1}} \left(C_0(\alpha) + C_1(\alpha)\;\frac{12\ell^2+12\ell-2\alpha+1}{(4n+2\ell+3)^2} + O(1/n^4)\right)$$

- Numerical procedure: M equations for $E_0, \ldots E_{M-1}$,

$$Q(E) \approx const\; E^{\ell+\frac{1}{2}} \prod_{n=0}^{M-1} \big(1 - E/E_n\big) \prod_{n=M}^{\infty} \big(1 - E/E_n^{(\mathrm{WKB})}\big)$$

“Monster” potentials [3]

$$z \mapsto q\, z, \quad q = \mathrm{e}^{\frac{\pi \mathrm{i}}{\alpha+1}} : \quad -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + U_{\mathrm{eff}}(z) - E \mapsto \frac{1}{q^2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + q^2 U_{\mathrm{eff}}(qz) - q^2 E\right]$$

- Symmetry: $U_{\mathrm{eff}}(qz) = q^{-2}\, U_{\mathrm{eff}}(z)$
- Asymptotic behavior:

$$U_{\mathrm{eff}}(z) \to \begin{cases} \frac{\ell(\ell+1)}{z^2} + o(1) & \text{as} \quad z \to 0 \\ z^{2\alpha} + o(1) & \text{as} \quad z \to \infty \end{cases}$$

- For any E all solutions of $\big(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + U_{\mathrm{eff}}(z) - E\big)\psi = 0$ are monodromy free everywhere (so that only poles are allowed, no branch points) except for $z = 0$ and $z = \infty$.

$$U_{\mathrm{eff}}(z) = \frac{\ell(\ell+1)}{z^2} + z^{2\alpha} - 2\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sum_{k=1}^{L} \log\big(z^{2\alpha+2} - z_k\big)\ ,$$

where $z_k,\ k = 1, 2 \ldots L$ satisfies a certain system of L algebraic equations.

$$\boxed{\frac{1}{2\pi \mathrm{i}}\ \log\left(\frac{Q(q^{-2}E_n)}{Q(q^2 E_n)}\right) = N_n + \frac{1}{2} \qquad (N_n \in \mathbb{Z})}$$

Monster potential $\leftrightarrow$ set of integers $\{N_n\}$.

Exercises

Exercise I.1. Show that the change of variables $z=\mathrm{e}^{y}, \quad \psi=\mathrm{e}^{\frac{y}{2}}\ \tilde{\psi}$ brings the

$$\Big[-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{\ell(\ell+1)}{z^2}+z^{2\alpha}\Big]\ \psi=E\,\psi$$

to the form

$$\Big[-\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\mathrm{e}^{2(1+\alpha)y}-E\,\mathrm{e}^{2y}\Big]\ \tilde{\psi}=-4k^2\ \tilde{\psi}\ , \qquad \text{where} \quad k=\tfrac{1}{4}\ (2\ell+1)\ .$$

Exercise I.2. Let us define the functions

$$A_{\pm}(E)\equiv\frac{W_{\pm}(E,\ell)}{W_{\pm}(0,\ell)}$$

and

$$f(z,E)=\sqrt{-4k^2+Ez-z^{1+\alpha}} \qquad \big(k=\tfrac{1}{4}\ (2\ell+1)\big).$$

In order to make $f(z,E)$ a single-valued function of the variable z, we introduce a branch cut along the segment $z\in[0,z^*],\ f(z^*)=0$ and set $f(z+\mathrm{i}0)>0,\ z\in[0,z^*]$.[1]

Show that in the WKB approximation

$$A_{\pm}(E)\approx\exp\bigg(\mathrm{i}\int_{C_{\pm}}\frac{\mathrm{d}z}{2z}\ \big(f(z,E)-f(z,0)\big)\bigg)\ ,$$

where the contour $C_+(C_-)$ starts at the point $z=0$, goes below (above) the cut and then to $z\to+\infty$. In this case

$$q^{2\ell+1}\ \frac{A_+(Eq^2)}{A_+(Eq^{-2})}\approx\exp\bigg(-\mathrm{i}\int_{C}\frac{\mathrm{d}z}{2z}\ f(z,E)\bigg)\ ,$$

here the contour C starts at $z=z^*$ above the cut, goes around the segment $[0,z^*]$ and returns to $z=z^*$. At the same time,

$$\frac{A_+(E)}{A_-(E)}\approx\exp\bigg(-\mathrm{i}\int_{\tilde{C}}\frac{\mathrm{d}z}{2z}\ \big(f(z,E)-f(z,0)\big)\bigg)\ ,$$

here the contour $\tilde{C}$ starts at $z=0$ below the cut, goes around the segment $[0,z^*]$ and returns to the $z=0$.

Exercise I.3. (a) Using the result of the previous exercise show that for $\alpha>1$

$$A_+(E)=(-E)^{-k}\,D_0^{-1}\ \exp\bigg(\frac{\pi}{\cos(\frac{\pi}{2\alpha})}\ (-E/C_0)^{\frac{\alpha+1}{2\alpha}}+o(1)\bigg) \quad \text{as} \quad |\Im m(-E)|<\pi\ , \quad E\to\infty\ ,$$

where

$$C_0=\bigg[\frac{2\sqrt{\pi}\Gamma(\frac{3}{2}+\frac{1}{2\alpha})}{\Gamma(1+\frac{1}{2\alpha})}\bigg]^{\frac{2\alpha}{\alpha+1}}\ .$$

[1] Here, for simplicity, we assume that $\Re e\big(\ell+\frac{1}{2}\big)=0$ and $\Im m(E)=0$.

The E-independent constant D_0 remains undetermined within the WKB approximation. Notice that it can be naturally identified with the (regularized) functional determinant

$$D_0 = \mathrm{Det}^{(\mathrm{reg})}(\hat{H}) \ , \qquad \hat{H} = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\ell(\ell+1)}{z^2} + z^{2\alpha} \ .$$

(b) Derive the BS quantization condition:

$$E_n \approx C_0 \, \big(n + \tfrac{1}{4}(2\ell+3)\big)^{\frac{2\alpha}{\alpha+1}} \qquad (n \gg 1) \ .$$

Exercise I.4. (a) Show that, in the case $\alpha = 1$,

$$A_\pm(E) \equiv \frac{W_\pm(E,\ell)}{W_\pm(0,\ell)} = \frac{\Gamma(\frac{1}{2} \pm k) \ \mathrm{e}^{\frac{E}{4}\gamma_E}}{\Gamma(\frac{1}{2} \pm k - \frac{E}{4})}$$

where $\gamma_E = 0.5772\ldots$ stands for the Euler constant and $k = \frac{1}{4}\,(2\ell+1)$.

(b) Show that for $\alpha \to \infty$

$$\lim_{\alpha\to\infty} A_\pm(E) = \Gamma(1 \pm 2k) \ (\sqrt{E}/2)^{\mp 2k} \ J_{\pm 2k}(\sqrt{E}) \ ,$$

where $J_\nu(z)$ is the conventional Bessel function.

Exercise I.5. Given the spectral set $\{E_n\}_{n=0}^{\infty}$ it is useful to introduce the following function

$$\Theta_k(\omega) = \frac{\sqrt{\pi} \ 2^{1+\mathrm{i}\omega} \ \Gamma(\frac{\mathrm{i}(1+\alpha)\omega}{2\alpha})}{\Gamma(\frac{\mathrm{i}\omega}{2\alpha})\Gamma(-\frac{1}{2} + \frac{\mathrm{i}\omega}{2})} \ \sum_{n=0}^{\infty} (E_n)^{-\frac{\mathrm{i}\omega(1+\alpha)}{2\alpha}} \ ,$$

where

$$2k = \ell + \tfrac{1}{2} \ .$$

(a) Show that $\Theta_k(\omega)$ is an analytic function in the half-plane $\Im m(\omega) < -1$ and

$$\log A_+(E) = -\frac{\mathrm{i}}{4\pi^{\frac{3}{2}}} \int_{C_\omega} \frac{\mathrm{d}\omega}{\omega} \ \Gamma(1 - \tfrac{\mathrm{i}\omega(1+\alpha)}{2\alpha})\Gamma(\tfrac{\mathrm{i}\omega}{2\alpha})\Gamma(\tfrac{\mathrm{i}\omega - 1}{2}) \ 2^{-\mathrm{i}\omega} \ \ \Theta_k(\omega) \ .$$

Here the integration contour goes along the line $\Im m(\omega) = -1-\epsilon$ with arbitrary small $\epsilon > 0$.

(b) Using the result of Exercise I.2, show that for any $\Im m(\omega) < -1$

$$\Theta_k(\omega) = (1+\alpha)^{-1} \ k^{1-\mathrm{i}\omega} \ \big(1 + o(1)\big) \qquad \text{as} \qquad k \to +\infty \ .$$

(c) Show that for $\alpha = 1$, $\Theta_k(\omega)$ is an entire function of ω.

Hint: Check that in this case $\Theta_k(\omega)$ can be expressed in terms of the Hurwitz ζ-function $\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}$:

$$\Theta_k(\omega) = 2^{2\mathrm{i}\omega - 1} \ (\mathrm{i}\omega - 1) \ \zeta\big(\mathrm{i}\omega, k + \tfrac{1}{2}\big) \ .$$

Excercise* **I.6**. Show that $\Theta_k(\omega)$ is an entire function of ω for any $\alpha > 0$.

Hint: The proof is based on the so called DDV equation (see Appendix A in ref. [4]).

Exercise I.7. Using the result of Exercise I.7, show that for $|\Im m(-E)| < \pi$, $E \to \infty$,

$$A_+(E) = D_0^{-1}\,(-E)^{-k}\ \exp\left(\frac{\pi}{\cos(\frac{\pi}{2\alpha})}\,(-E/C_0)^{\frac{\alpha+1}{2\alpha}} + C_{-1}\,(-E/C_0)^{-\frac{\alpha+1}{2\alpha}} + o\big(E^{-\frac{\alpha+1}{2\alpha}}\big)\right),$$

where the constant C_0 is the same as in Exercise I.3, whereas

$$\begin{aligned}\log D_0 &= -\frac{2\alpha k}{1+\alpha}\,\Big[\log(4/\mathrm{e}) + \mathrm{i}\,\partial_\omega \log\Theta_k(0)\Big]\\ C_{-1} &= -\frac{\Theta_k(\mathrm{i})}{\sin(\frac{\pi}{2\alpha})}\,.\end{aligned}$$

Notice that the constant is the so-called zeta-regularized functional determinant

$$D_0 = \mathrm{Det}^{(\mathrm{reg})}(\hat{H})\,, \qquad \hat{H} = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\ell(\ell+1)}{z^2} + z^{2\alpha}\,.$$

Excercise I.8. Show that[2]

(a) $\Theta_k(0) = \frac{1}{1+\alpha}\,k$

(b) $\mathrm{i}\,\partial_\omega \log\Theta_k(0) = \frac{1+\alpha}{2\alpha k}\,\log\left(\frac{1}{\sqrt{1+\alpha}}\,\frac{\Gamma(1+2k)}{\Gamma(1+\frac{2k}{1+\alpha})}\right) - \frac{1}{\alpha}\,\log\big((1+\alpha)(2/\mathrm{e})^\alpha\big)$

(c) $\Theta_k(\mathrm{i}) = -\frac{1}{24} + \frac{k^2}{1+\alpha}$.

(d) $\Theta_k\big(\mathrm{i}(2m-1)\big) = \frac{1}{1+\alpha}\,P_m(k^2)\quad (m = 1,\,2,\,3\ldots)$, where P_m is a polynomial of degree m in k^2 such that $P_m(k^2) = k^{2m} - \frac{m(2m-1)}{24}\,(1+\alpha)\,k^{2m-2} + \ldots$.

Excercise I.9. The substitution $u = 2(1+\alpha)y - 2\log 2(1+\alpha)$ brings

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \mathrm{e}^{2(1+\alpha)y} - E\,\mathrm{e}^{2y}\right]\tilde{\psi} = -4k^2\,\tilde{\psi}$$

to the form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \mathrm{e}^u + \delta U(u)\right]\tilde{\psi} = -\rho^2\,\tilde{\psi}\,, \quad \text{where}\ \ \delta U(u) = -E\,\big(2+2\alpha\big)^{-\frac{2\alpha}{1+\alpha}}\,\mathrm{e}^{\frac{u}{1+\alpha}}\,, \qquad \rho = \frac{k}{1+\alpha}\,.$$

Consider the Lipman-Schwinger equation

$$\chi(u) = K_{2\rho}(\mathrm{e}^u) - \int_{-\infty}^{\infty}\mathrm{d}u'\ G(u,u')\,\chi(u')\,,$$

where $G(u,u')$ is the Green's function for the last ODE with $\delta U = 0$ subject to the asymptotic condition $\lim_{u\to\infty} G(u,u') = 0$.

(a) Show that

$$\frac{1}{A_+(E)} = 1 - \int_{-\infty}^{\infty}\mathrm{d}u\ I_{2\rho}\big(\mathrm{e}^u\big)\ \delta U(u)\ \chi(u)\,.$$

[2] If you cannot prove these equations, please check them for the case $\alpha = 1$.

(b) Calculate the value $\Theta(-\frac{2\mathrm{i}\alpha}{1+\alpha})$.
Here $K_\rho(z)$ and $I_\rho(z)$ denote the conventional modified Bessel functions.

Excercise I.10. Let $\{z_j\}_{j=1}^{L}$ be a set of *complex* numbers satisfying the system of L algebraic equations

$$\sum_{\substack{m=1\\ m\neq j}}^{L} \frac{z_j\big(z_j^2+(3+\alpha)(1+2\alpha)z_j z_m+\alpha(1+2\alpha)z_m^2\big)}{(z_j-z_m)^3}-\frac{\alpha\, z_j}{4(1+\alpha)}+\Delta=0\ ,$$

where $\Delta=\frac{(2\ell+1)^2-4\alpha^2}{16(\alpha+1)}$. Show that all solutions of the ODE

$$\big(-\tfrac{\mathrm{d}^2}{\mathrm{d}z^2}+U_{\rm eff}(z)-E\big)\psi=0$$

with

$$U_{\rm eff}(z)=\frac{\ell(\ell+1)}{z^2}+z^{2\alpha}-2\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sum_{k=1}^{L}\log\big(z^{2\alpha+2}-z_k\big)\ ,$$

are monodromy free everywhere except for $z=0$ and $z=\infty$.

Hint: See Appendix B in ref. [4].

References

[1] L. D. Landau and E. M. Lifshitz, "Quantum Mechanics" (Volume 3 of A Course of Theoretical Physics)

[2] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, J. Statist. Phys. **102**, 567 (2001) [arXiv:hep-th/9812247].

[3] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, Adv. Theor. Math. Phys. **7**, 711 (2003) [arXiv:hep-th/0307108].

[4] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, Commun. Math. Phys. **190**, 247 (1997) [arXiv:hep-th/9604044].