Lecture III: The Thirring Model

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Jordan–Wigner transformation

The goal of this lecture is to better understand the scaling limit. To do this, start with the Heisenberg XXZ model. The Heisenberg XXZ model describes a system of N one dimensional spins on the line with nearest neighbor interactions. The Hamiltonian is:

$$\boldsymbol{H}_{XXZ} = -J\sum_{m=1}^{N} \left(\sigma^x_m\,\sigma^x_{m+1} + \sigma^y_m\,\sigma^y_{m+1} + \Delta\,\sigma^z_m\,\sigma^z_{m+1}\right), \tag{1}$$

where the Pauli Matrices σ^a_m act on the m^{th} site. The quasiperiodic boundary conditions are:

$$\sigma^{\pm}_{N+1} = \mathrm{e}^{\pm 2\mathrm{i}\theta}\,\sigma^{\pm}_1\ . \tag{2}$$

The Hamiltonian commutes with the total spin operator $\boldsymbol{S}^z$:

$$\boldsymbol{S^z} = \frac{1}{2}\,\sum_{m=1}^{N}\sigma^z_m\,,$$

so that the total number of up spins n is conserved by the Hamiltonian. The Heisenberg spin chain can be converted to a system of fermions by the Jordan–Wigner transformation.

The Pauli Matrices at a particular site satisfy the fermionic anti–commutation relations:

$$\{\sigma^+_m,\,\sigma^+_m\} = 0\,,\qquad \{\sigma^-_m,\,\sigma^-_m\} = 0\,,\qquad \{\sigma^+_m,\,\sigma^-_m\} = 1\,. \tag{3}$$

In other words, the $\sigma^{\pm}_m$ behave as:

$$\sigma^+_m \sim \psi^{\dagger}_m\,,\qquad\qquad \sigma^-_m \sim \psi_m\,,$$

where the vector space at site i, $\mathbb{C}^2$ is spanned by the basis vectors:

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix},\qquad\qquad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

However, the Pauli matrices at different sites do not anticommute. In order to really consider the system of spins as fermions, it is necessary to multiply the Pauli matrices by some operator, C_m, so that the anti–commutation relations (3) are preserved and the Pauli matrices at different sites anti–commute. A simple way to satisfy this would be if the C_m commute with σ^a_j for $j \geq m$ and anti–commute with σ^a_j for $j < m$. For two spins:

$$\Psi^{\dagger}_1 = \sigma^+_1\,,\qquad\qquad \Psi_1 = \sigma^-_1\,,$$

$$\Psi^{\dagger}_2 = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\sigma^z_1}\,\sigma^+_2\,,\qquad\qquad \Psi_2 = \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\sigma^z_1}\,\sigma^+_2\,,$$

where $e^{\frac{i\pi}{2}\sigma_1^z} = i\sigma_1^z$. The generalization to N spins is trivial:

$$\begin{aligned}\Psi_m^\dagger &= \exp\Big(-\frac{i\pi}{2}\sum_{j<m}\sigma_j^z\Big)\,\sigma_m^+ \\ \Psi_m &= \exp\Big(+\frac{i\pi}{2}\sum_{j<m}\sigma_j^z\Big)\,\sigma_m^-\end{aligned}$$

which can be equivalently written as:

$$\begin{aligned}\sigma_m^+ &= \exp\Big(+i\pi\sum_{j<m}\big(\Psi_j^\dagger\,\Psi_j-\tfrac{1}{2}\big)\Big)\,\Psi_m^\dagger \\ \sigma_m^- &= \exp\Big(-i\pi\sum_{j<m}\big(\Psi_j^\dagger\,\Psi_j-\tfrac{1}{2}\big)\Big)\,\Psi_m\end{aligned}$$

due to the identity: $\sigma^z = 2\psi^\dagger\psi - 1$.

Exercise III.1: Show that the Jordan–Wigner transformation maps the Hamiltonian XXZ-spin chain (1),(2) to

$$\boldsymbol{H}_{XXZ} = -2J\sum_{m=1}^{N}\Big(i\big(\Psi_m^\dagger\Psi_{m+1}-\Psi_{m+1}^\dagger\,\Psi_m\big) + 2\Delta\,\big(\Psi_m^\dagger\Psi_m-\tfrac{1}{2}\big)\big(\Psi_{m+1}^\dagger\Psi_{m+1}-\tfrac{1}{2}\big)\Big)\,,$$

where the fermions satisfy the Boundary Conditions (BC)

$$\Psi_{N+1}^\dagger = -e^{-i\pi S^z+2i\theta}\,\Psi_1^\dagger\,,\quad \psi_{N+1} = -\Psi_1\,e^{+i\pi S^z-2i\theta}\,. \tag{4}$$

Low energy effective theory for $\Delta = 0$

Clearly it make sense to start with the case $\Delta = 0$, where the Hamiltonian describes a system of free fermions and can be easily diagonalized. For simplicity, let us assume that that N is an even number, i.e., S^z is an integer. Apply the Fourier transform

$$\begin{aligned}\Psi_m &= \frac{i^{-m}}{\sqrt{N}}\sum_{j=-M+1}^{N-M}\mathsf{c}_j\,e^{im\mathrm{k}_j} \\ \Psi_m^\dagger &= \frac{i^{+m}}{\sqrt{N}}\sum_{j=-M+1}^{N-M}\mathsf{c}_j^\dagger\,e^{-im\mathrm{k}_j}\,.\end{aligned}$$

Here M stands for the integer part of $\frac{N}{4}$. The expansion coefficients c_j satisfy the anti–commutation relations:

$$\{\mathsf{c}_m^\dagger,\mathsf{c}_l\} = \delta_{m,l} \qquad \{\mathsf{c}_m^\dagger,\mathsf{c}_l^\dagger\} = \{\mathsf{c}_m,\mathsf{c}_l\} = 0\,.$$

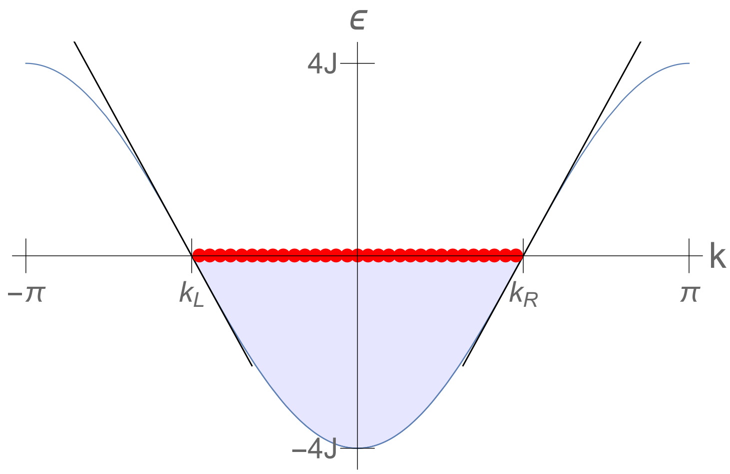


Figure 1: A plot of the dispersion relation $\epsilon = -4J\cos(\mathtt{k})$ for the first Brillouin zone.

Admissible values for the quasimomentum $\mathtt{k}$ follow from the twisted BC (4):

$$\mathtt{k}_j = \tfrac{\pi}{2} - \tfrac{2\pi}{N}\Big(j - \tfrac{1}{2} + \big(\tfrac{\theta}{\pi} + \tfrac{1}{2}\,S^z\big)\Big) \ . \tag{5}$$

Substituting this into the Hamiltonian $\boldsymbol{H}_{XXZ}$ with $\Delta = 0$ yields:

$$H \ = \sum_j \epsilon(\mathtt{k}_j)\ \mathtt{c}_j^\dagger\,\mathtt{c}_j\,, \qquad \text{where} \qquad \epsilon(\mathtt{k}) \ = \ -4J\cos(\mathtt{k})\,. \tag{6}$$

The operators $\mathtt{c}_j$ should be thought of as the creation and annihilation operators of a particle with definite momentum $\mathtt{k}_j$ and with the dispersion relation $\epsilon = \epsilon(\mathtt{k})$. A plot of the dispersion relation is given in figure 1.

For quasimomentum $\mathtt{k}_L \leq \mathtt{k}_j \leq \mathtt{k}_R$,

$$\mathtt{k}_L = -\tfrac{\pi}{2} + \tfrac{2\pi}{N}\Big(\tfrac{1}{2} - \tfrac{\theta}{\pi} + \tfrac{1}{2}\,S^z\Big)\ , \quad \mathtt{k}_R = +\tfrac{\pi}{2} - \tfrac{2\pi}{N}\Big(\tfrac{1}{2} + \tfrac{\theta}{\pi} + \tfrac{1}{2}\,S^z\Big)\ , \tag{7}$$

the energy of the particle is negative. In the ground state, these energy levels are filled and form a Fermi sea, so that the vacuum energy is given by

$$E^{(vac)} = -4J\sum_{j=1}^{\frac{N}{2}-S^z}\cos(\mathtt{k}_j)\ . \tag{8}$$

Exercise III.2. Show that, as N goes to infinity,

$$E^{(vac)} = -\frac{4JN}{\pi} - 4J\ \frac{\pi}{6N}\ \Big(1 - 12\,\theta^2 - 3\,(S^z)^2\Big) + O(N^{-3})\ . \tag{9}$$

The excited energy states are particles of positive energy or holes in the Fermi sea. As an illustration, let us focus on the case $S^z = \theta = 0$. Introduce the notations

$$\begin{aligned} \mathtt{a}_{\frac{1}{2}-j} &= \mathtt{c}_j\ , \qquad && -M+1 \leq j \leq 0 \\ \mathtt{a}_{j-\frac{1}{2}} &= \mathtt{c}_j^\dagger\ , \qquad && 1 \leq j \leq M \end{aligned}$$

and

$$\begin{aligned} \mathtt{a}_{j-\frac{1}{2}} &= \mathtt{c}^{\dagger}_{\frac{N}{2}+j} \ , \qquad M-\tfrac{N}{2}<j\leq 0 \\ \mathtt{b}_{\frac{1}{2}-j} &= \mathtt{c}_{\frac{N}{2}+j} \ , \qquad 1\leq j\leq \tfrac{N}{2}-M \end{aligned}$$

(recall that $M=[N/4]$). The low energy physics is described by the low energy hole and particle excitations that lie close to the "Fermi surface" $\mathtt{k}\in\{\mathtt{k}_R,\,\mathtt{k}_L\}$, where the dispersion is approximately linear:

$$\epsilon = 4J\sin(\tfrac{2|\nu|\pi}{N}) \approx \tfrac{8\pi J}{N}\ |\nu| \ , \qquad \nu=\pm\tfrac{1}{2},\ ,\pm\tfrac{3}{2}\ldots\ \ll N \ .$$

The dynamics of the low-energy excitations is captured by the effective Hamiltonian:

$$\boldsymbol{H}_{XXZ} = -\frac{4JN}{\pi}+\frac{8\pi J}{N}\left[-\frac{1}{12}+\sum_{\nu=\pm\frac{1}{2},\pm\frac{3}{2}\ldots}^{\infty}|\nu|\left(\mathtt{a}^{\dagger}_{\nu}\,\mathtt{a}_{\nu}\,+\,\mathtt{b}^{\dagger}_{\nu}\,\mathtt{b}_{\nu}\right)\right]+O(N^{-3}) \ .$$

Notice that the parameter

$$a=(4J)^{-1}$$

has the dimension of length and can be interpreted as a dimensionful lattice spacing. The scaling limit is $a\to 0$, $N\to\infty$ while the macroscopic length of the system,

$$R=aN \ ,$$

is kept fixed. The scaling limit lead to a low energy effective Hamiltonian

$$\boldsymbol{H}_{\mathrm{eff}} = -\frac{1}{\pi}\,\frac{R}{a^2}+\frac{2\pi}{R}\left[-\frac{1}{12}+\sum_{\nu=\pm\frac{1}{2},\pm\frac{3}{2}\ldots}^{\infty}|\nu|\left(\mathtt{a}^{\dagger}_{\nu}\,\mathtt{a}_{\nu}\,+\,\mathtt{b}^{\dagger}_{\nu}\,\mathtt{b}_{\nu}\right)\right] \ , \tag{10}$$

which describes the physics of the system at energy scales significantly below that of the microscopic energy scale J. The first term here diverges as $a\to 0$. It is proportional to the size of the system and represents the contribution of the modes whose energy exceed the ultraviolet (UV) cut-off scale $\Lambda_{UV}=a^{-1}$. This "quadratic" divergency is absorbed by the so-called counterterm of the identity operator and is ignored in the context of QFT. The term $-\frac{\pi c}{6R}$ with $c=1$ is somewhat universal and can be interpreted as the Casimir energy. The spectrum of the effective Hamiltonian is given by

$$E_{\mathrm{eff}} = -\frac{R}{a^2}+\frac{2\pi}{R}\left[-\frac{1}{12}+\frac{1}{2}\,\left(L+\bar{L}\right)\right] \ , \qquad L,\,\bar{L}=0,\,1,\,2,\ldots \ .$$

The nonnegative integers L and $\bar{L}$ represent the contributions of the modes with half-integers $\nu>0$ and $\nu<0$, respectively.

In the scaling limit the lattice fermions are approximated as

$$\begin{aligned}\Psi_m &\approx \sqrt{a}\,\Big[\,\mathrm{i}^{-m}\,\psi_R(x^1)+\mathrm{i}^{+m}\,\psi^\dagger_L(x^1)\,\Big]\\ \Psi^\dagger_m &\approx \sqrt{a}\,\Big[\,\mathrm{i}^{+m}\,\psi^\dagger_R(x^1)+\mathrm{i}^{-m}\,\psi_L(x^1)\,\Big]\,,\end{aligned} \tag{11}$$

where $x^1 = ma$ and the continuous Fermi fields

$$\begin{aligned}\psi_R(x^1) &= R^{-\frac{1}{2}}\sum_{\nu=+\frac{1}{2},+\frac{3}{2},\ldots}\Big(\mathsf{a}_\nu\,\mathrm{e}^{2\pi\mathrm{i}\nu\,\frac{x^1}{R}}+\mathsf{b}^\dagger_\nu\,\mathrm{e}^{-2\pi\mathrm{i}\nu\,\frac{x^1}{R}}\Big)\\ \psi_L(x^1) &= R^{-\frac{1}{2}}\sum_{\nu=-\frac{1}{2},-\frac{3}{2},\ldots}\Big(\mathsf{a}_\nu\,\mathrm{e}^{2\pi\mathrm{i}\nu\,\frac{x^1}{R}}+\mathsf{b}^\dagger_\nu\,\mathrm{e}^{-2\pi\mathrm{i}\nu\,\frac{x^1}{R}}\Big)\end{aligned}$$

obey the canonical anticommutation relations

$$\begin{aligned}&\{\psi^\dagger_R(x^1),\,\psi_R(y^1)\} = \{\psi^\dagger_R(^1x),\,\psi_R(y^1)\} = \delta(x^1-y^1)\,,\\ &\{\psi_{R,L}(x^1),\,\psi_{R,L}(y^1)\} = \{\psi^\dagger_{R,L}(x^1),\,\psi^\dagger_{R,L}(y^1)\} = 0\end{aligned} \tag{12}$$

and the antiperiodic boundary conditions[1]

$$\psi_\pm(x^1+R) = -\psi_\pm(x^1)\,.$$

Notice that here we use the QFT notation x^1 for the space coordinate. The time variable will be denoted below by x^0.

The low energy effective Hamiltonian can be expressed in terms of $\psi_{R,L}$.

Exercise III.3.: Show that, ignoring the quadratic divergency, the Hamiltonian (10) can be written in the form

$$\begin{aligned}\boldsymbol{H}_{\rm eff} &= \int_0^R \mathrm{d}x^1\;\mathcal{H}_0(x)\,,\\ \mathcal{H}_0(x) &= -\frac{\mathrm{i}}{2}\,\big(\psi^\dagger_R\partial_1\psi_R+\psi_R\partial_1\psi^\dagger_R-\psi^\dagger_L\partial_1\psi_L-\psi_L\partial_1\psi^\dagger_L\big)\qquad \big(\partial_1=\tfrac{\partial}{\partial x^1}\big)\,.\end{aligned} \tag{13}$$

Hint: In the analytical regularization the divergent sum $\frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\ldots$ is replaced by the Hurwitz zeta function $\zeta\big(-1,\frac{1}{2}\big)=\frac{1}{24}$.

In the Lagrangian description both the canonical (anti)commutation relations and Hamiltonian are encoded by means of the Lagrangian.

Exercise III.4.: Show that the Lagrangian density corresponding to the canonical anticommutation relations (12) and the Hamiltonian (13) is given by

$$\mathcal{L}_0(x) = \mathrm{i}\,\big(\psi^\dagger_R\partial_+\psi_R+\psi_R\partial_+\psi^\dagger_R+\psi^\dagger_L\partial_-\psi_L+\psi_L\partial_-\psi^\dagger_L\big)\,,\quad\text{where}\qquad \partial_\pm=\tfrac{1}{2}\,(\partial_0\pm\partial_1)\,. \tag{14}$$

[1]Recall that we are focusing here on the scaling limit of the XXZ spin chain with $\Delta = S^z = \theta = 0$.

Notice that $\mathcal{L}_0(x)$ is a local function built from the local fermionic fields $\psi_{R,L}(x)$ taken at the same space-time point $x = (x^0, x^1)$.

The classical equations of motion corresponding to the Lagrangian density (14) read as

$$\partial_+\psi_R \,=\, 0 \qquad\qquad \partial_-\psi_L \,=\, 0 \ . \tag{15}$$

In the case of infinite volume, i.e. $R = \infty$, a general solution of these equation represent a traveling wave propagating with a constant speed either to the right $\big(\psi_R = \psi_R(x^0 - x^1)\big)$, or to the left $\big(\psi_R = \psi_R(x^0 + x^1)\big)$.[2] Notice that there is a freedom in the definition of the lattice spacing $a \propto J^{-1}$. In the above considerations, we set the proportionality coefficient in the relation between a and J^{-1} as $\frac{1}{4}$. With this choice, the "speed of light" turns out to be one. This is a standard convention used in QFT.

Low energy effective theory for $-1 < \Delta < 1$

We now turn to the description of the low energy physics for the general XXZ model in the disordered phase. In our analyses, in order to avoid some technical complexification related to the finite size effects, it is useful to start with the infinite size system, where not only $N \to \infty$ but also the macroscopic size R is chosen to be infinite.

We shall use a powerful physical technique which is known as the Renormalization Group (RG) approach. In fact, it can be applied to many lattice systems. Its starting point is based on finding the so-called RG fixed point – the particular point in the space of parameters of the lattice system where the correlation length turns to be zero. Roughly speaking all the low energy excitations in this case are "photons" or "neutrinos" – massless particles which are either bosons or fermions moving at light speed. The point $\Delta = 0$ is an example of such a RG fixed point. The low energy behavior at the RG fixed point is described by a special class of QFT, the so-called Conformal Field Theory (CFT). Then one can consider the vicinity of the RG-fixed point by means of a sort of perturbation theory where the corresponding Lagrangian density is represented by the form

$$\mathcal{L}_{\text{pert}}(x) = \mathcal{L}_0(x) + \sum_j \mathsf{g}_j\, \mathcal{O}_j(x) \ , \tag{16}$$

where $\{\mathcal{O}_j\}$ is the set of all possible local fields while $\{\mathsf{g}_j\}$ are the corresponding coupling constants. Remarkably, if we interesting in the low energy behavior of the system, there is no need to consider the infinite number of terms in this sum. Only contributions of the so-called relevant and marginal local fields affect the low energy behavior. The scaling dimension $d_{\mathcal{O}}$ of the relevant (marginal) operator $\mathcal{O}$ is smaller than (equal) the space-time dimension D. As an illustration of the notion of the scaling dimension, let us note that the scaling dimension of the local fermions fields d_ψ is equal to $\frac{1}{2}$. This can be seen from the formulae (11) – the lattice fermions are, of course, dimensionless quantities, so the

[2]Such fields are sometime referred to as "chiral fields".

overall factor $a^{\frac{1}{2}}$ sets the dimensions of the local fermion fields. There are only a finite number of local fields that can be built from $\psi_R,\ \psi_R^\dagger$ and $\psi_L,\ \psi_L^\dagger$ with scaling dimensions lower or equal to $D=2$. For example,

$$\psi_R^\dagger \psi_R\ , \quad \psi_R^\dagger \partial_\pm \psi_R\ , \quad \psi_R^\dagger \psi_L^\dagger \psi_R \psi_L\ , \ldots\ . \tag{17}$$

In fact, not all possible local fields can appear in the sum (16). Some of them are prohibited by the exact global symmetries of the lattice system. In the case under consideration, the following symmetries of the XXZ spin chain can be easily established:

- The infinite XXZ spin chain ($N=\infty$) is evidently invariant with respect to the parity transformation $\mathbb{P}$,

$$\mathbb{P}\,\sigma_m^a\,\mathbb{P} = \sigma_{-m}^a \qquad (\mathbb{P}^2 = \mathbb{I})\ ,$$

  where $a=\pm, z$.

- The symmetry group of the chain contains also the "charge conjugation" $\mathbb{C}$:

$$\mathbb{C}\,\sigma_m^\pm\,\mathbb{C} = \sigma_m^\mp\,, \qquad \mathbb{C}\,\sigma_m^z\,\mathbb{C} = -\sigma_m^z \qquad\qquad (\mathbb{C}^2 = \mathbb{I})\ .$$

- The time reversal symmetry acts as

$$\mathbb{T}\,\sigma_l^\pm\,\mathbb{T} = \sigma_l^\mp\ , \qquad\quad \mathbb{T}\,\sigma_l^z\,\mathbb{T} = -\sigma_l^z\ .$$

  Notice that the reversal is an anti-unitary transformation, so that $\mathbb{C}$ and $\mathbb{T}$ correspond to different symmetries of the XXZ chain, even thought they act identically on the spin operators.

- The infinite XXZ chain is invariant w.r.t.

$$\mathbb{K}\,\sigma_l^a\,\mathbb{K}^{-1} = \sigma_{l+1}^a\ .$$

- Finally, the chain is invariant under continuous $U(1)$ rotations,

$$\mathbb{U}_\alpha\,\sigma_m^\pm\,\mathbb{U}_\alpha^{-1} = \mathrm{e}^{\pm i\alpha}\ \sigma_m^\pm\,, \qquad \mathbb{U}_\alpha\,\sigma_m^z\,\mathbb{U}_\alpha^{-1} = \sigma_m^z$$

  infinitesimally generated by the

$$\boldsymbol{S}^z = \frac{1}{2}\,\sum_l \sigma_l^z\ : \qquad \mathbb{U}_\alpha = \mathrm{e}^{\mathrm{i}\alpha \boldsymbol{S}^z}\ .$$

Th above lattice symmetries act on on the continuous Fermi fields in the following way:

$$\begin{aligned}
\mathbb{C}\psi_L(t,x)\mathbb{C} &= \psi_L^\dagger(t,x)\ , \qquad \mathbb{C}\psi_R(t,x)\mathbb{C} = \psi_R^\dagger(t,x)\\
\mathbb{P}\psi_R(t,x)\mathbb{P} &= \psi_L(t,-x)\ , \quad\ \mathbb{P}\psi_L(t,x)\mathbb{P} = \psi_R(t,-x)\\
\mathbb{T}\psi_L(t,x)\mathbb{T} &= \psi_R^\dagger(-t,x)\ , \quad\ \mathbb{T}\psi_R(t,x)\mathbb{T} = \psi_L^\dagger(-t,x)
\end{aligned}$$

and

$$\mathbb{U}_\alpha\,\psi_R\,\mathbb{U}_\alpha^{-1} = \mathrm{e}^{\mathrm{i}\alpha}\ \psi_R\ , \qquad \mathbb{U}_\alpha\,\psi_L\,\mathbb{U}_\alpha^{-1} = \mathrm{e}^{-\mathrm{i}\alpha}\ \psi_L\ .$$

It is easy to see from (11) that the lattice translations act on the continuous Fermi fields in the following way:

$$\mathbb{K}\,\psi_R(x)\,\mathbb{K}^{-1} = \mathrm{i}\,\psi_R(x+a)\,, \qquad \mathbb{K}\,\psi_L(x)\,\mathbb{K}^{-1} = \mathrm{i}\,\psi_L(x+a)\,.$$

Since in the scaling limit $a \to 0$, the Lagrangian density of the effective low energy theory should possess $\mathbb{Z}_4$ symmetry generated by the transformation

$$\psi_R \mapsto \mathrm{i}\,\psi_R\ , \qquad \psi_L \mapsto \mathrm{i}\,\psi_L\ .$$

Exercise III.5. Using the above global symmetries show that the low energy Lagrange density may contains only the following three real local fields with the scaling dimensions $d_{\mathcal{O}} \leq 2$: the free Hamiltonian density $\mathcal{H}_0(x)$ (13), the free Lagrangian density $\mathcal{L}_0(x)$ (14) and the marginal field

$$\mathcal{O}(x) = \psi_R^\dagger \psi_L^\dagger \psi_R \psi_L\ .$$

Hint. We do not need to include in the Lagrange density any total derivatives like, i.e., the fields like $\partial_\pm(\psi_R^\dagger\psi_R)$ and $\partial_\pm(\psi_L^\dagger\psi_L)$. Also notice that all the fields which contain powers of the Fermi fields taken at the same point, like $\psi_+^\dagger(\psi_+)^2\psi_-$, should be ignored. At the classical level they vanish identically. The corresponding quantum fields turn out to be irrelevant operators.

The effect of adding the fields $\mathcal{L}_0(x)$ and $\mathcal{H}_0(x)$ to the Lagrangian density $\mathcal{L}_0$ is somewhat trivial. These terms can be absorbed by the finite multiplicative renormalization of the Fermi fields,

$$\begin{aligned} \psi_R \mapsto \psi_- \equiv Z_\psi^{-\frac{1}{2}}\ \psi_R\,, \qquad & \psi_R^\dagger \mapsto \psi_-^\dagger \equiv Z_\psi^{-\frac{1}{2}}\ \psi_R^\dagger \\ \psi_L \mapsto \psi_+ \equiv Z_\psi^{-\frac{1}{2}}\ \psi_L\,, \qquad & \psi_L^\dagger \mapsto \psi_+^\dagger \equiv Z_\psi^{-\frac{1}{2}}\ \psi_L^\dagger\ , \end{aligned}$$

and the finite renormalization of the speed of light. The canonical light speed can be restored by adjusting the relation between the macroscopic energy scale J and the lattice spacing. Therefore, with a proper choice of the proportionality coefficient in the $J - a$ relation,

$$a = \mathcal{C}(\Delta)\ J^{-1}\ ,$$

the Lagrangian density describing the low energy behavior of the XXZ spin chain in the disorder regime is given by

$$\mathcal{L}(x) = \mathrm{i}\left(\psi_-^\dagger\partial_+\psi_- + \psi_-\partial_+\psi_-^\dagger + \psi_+^\dagger\partial_-\psi_- + \psi_+\partial_-\psi_-^\dagger\right) + 2\,\mathrm{g}_{4\mathrm{F}}\,\psi_+^\dagger\psi_-^\dagger\psi_+\psi_-\ .$$

Here g_{4F} is some real, dimensionless constant, depending on the value of the anisotropy parameter Δ.

The theory governed by the Lagrange density (18) is remarkable in many respects. First of all let us note that it possesses the Lorentz invariance. The Lorentz group is very special in $1+1$ Minkowski space. Under the Lorentz boost with rapidity θ, the linear combinations $x^{\pm} = x^0 \pm x^1$, transform irreducibly $x^{\pm} \mapsto \mathrm{e}^{\mp\theta}\, x^{\pm}$.

Exercise III.6. Explain the relation between the *WikipediA* definition of the Lorentz boost

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}\ , \qquad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{18}$$

and the transformation $(x')^{\pm} = \mathrm{e}^{\mp\theta}\, x^{\pm}$.

Assuming the the fields ψ_+ and ψ_- have Lorentz spins $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, i.e.,

$$\psi_{\pm} \mapsto \mathrm{e}^{\pm\frac{\theta}{2}}\, \psi_{\pm}\ ,$$

it is easy to establish the Lorentz invariance of $\mathcal{L}$. The Lagrange density can be brought to the conventional QFT form [2] by introducing the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ : \qquad \{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\ .$$

and the two-component Dirac field

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}\ , \qquad \bar{\psi} = \psi^{\dagger}\gamma^0\ .$$

Then $\mathcal{L}$ takes the form (up to a total derivative)

$$\mathcal{L} = \mathrm{i}\,\bar{\psi}\gamma^\mu\partial_\mu\psi - \tfrac{1}{2}\ \mathsf{g}_{4F}\ (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)\ .$$

The corresponding model is know as the (massless) Thirring model (1958). In the Condensed Matter community the model is known as Tomanaga (1950) – Luttinger (1963) model.

The four-fermion coupling g_{4F} is a dimensionless constant, i.e., the theory the Lagrangian density does not contain any dimensional parameter. It does not guarantee that the theory remains invariant at the quantum level. However,

Exercise III.7. Show that the β-function in the Thirring model vanishes at the lowest perturbative order.

The exact solution of the model, based on the so-called bosonization, shows that the model is a scale invariant QFT. In other words the whole interval $-1 < \Delta < 1$ corresponds to a one-parameter family of CFT.

The translational, Lorentz and scale invariance fix the form of the two-point function of the Dirac fields up to the overall constant.

Exercise III.8. Show that the normalization of the fermion fields can be chosen in such a way that

$$T\langle\, \psi(x)\, \bar{\psi}(0)\, \rangle = \frac{1}{2\pi \mathrm{i}}\ \frac{\gamma^\mu x_\mu}{(-x^2)^{\frac{1}{2}+d_\psi}}\ , \qquad \text{where} \qquad x^2 = x^\mu x_\mu - \mathrm{i}0\ .$$

Here $d_\psi = d_\psi(\Delta)$ stands for the scaling dimension.

Notice that, contrary to the case $\Delta = 0$, the Fermi fields $\psi_\pm$ are not chiral fields. Nevertheless the formula (11) can be generalized for non-vanishing anisotropy:

$$\begin{aligned}\Psi_m &\approx a^{d_\psi}\ \mathcal{A}_\psi\ \Big[\mathrm{e}^{-\mathrm{i}p_R x^1}\ \psi_-(x^1) + \mathrm{e}^{-\mathrm{i}p_L x^1}\ \psi_+^\dagger(x^1)\Big]\\ \Psi_m^\dagger &\approx a^{d_\psi}\ \mathcal{A}_\psi\ \Big[\mathrm{e}^{+\mathrm{i}p_R x^1}\ \psi_-^\dagger(x^1) + \mathrm{e}^{+\mathrm{i}p_L x^1}\ \psi_+(x^1)\Big]\ ,\end{aligned}$$

where the dimensionless amplitude $\mathcal{A}_\psi$ is some non-trivial function of the anisotropy. The scaling dimension d_ψ can be found explicitly from the exact BA solution. It reads explicitly as

$$d_\psi = \frac{1}{8g} + \frac{g}{2}\ , \qquad \text{where} \quad g = \frac{1}{\pi}\ \arccos(\Delta)\ .$$

To the best of my knowledge the calculation of the amplitude $\mathcal{A}_\psi$ remains an open interesting problem.

Finite size corrections to the XXZ energy spectrum

The above consideration allows one to make important prediction for the energy spectrum in the case $\Delta \neq 0$. In fact the qualitative structure of the spectra should remain the same as for the free fermion case. Namely, as $N \to \infty$,

$$E_{XXZ} = -\mathcal{E}_0\ \frac{R}{a^2} + E_{\text{Thirring}} + o(N^{-1})\ ,$$

where

$$R = Na\ , \qquad a = \mathcal{C}\ J^{-1}\ .$$

and

$$E_{\text{Thirring}} = \frac{2\pi}{R}\ \Big[-\frac{c_{\text{eff}}}{12} + \frac{1}{2}\ \big(L + \bar{L}\big)\Big]\ , \qquad L,\ \bar{L} = 0,\ 1,\ 2, \dots\ ,$$

All the dimensionless constants $\mathcal{E}_0$, $\mathcal{C}$ and c_{eff} depend on the anisotropy $\Delta = \cos(\pi g)$. It is most simple to calculate the ratio $\mathcal{E}_0/\mathcal{C}$:

Exercise III.9. Using the BA solution (in particular the result of Exercise II.7) show that

$$\mathcal{E}_0/\mathcal{C} = -\cos(\pi g) - 4\ \sin(\pi g) \int_0^\infty \frac{\mathrm{d}t}{\pi}\ \frac{\sinh(gt)}{\sinh(t)\cos((1-g)t)}\ .$$

One can also show that

$$\mathcal{C} = \frac{1-g}{2\sin(\pi g)}\ .$$

In fact, both $\mathcal{E}_0$ and $\mathcal{C}$ are the so-called "non-universal" constants. Their explicit values depend on the details of the microscopic Hamiltonian. The universal part of the energy is hidden in E_{Thirring}, and is controlled by the underlying CFT. Remarkably, all universal effects are encoded in the scaling part of the Bethe roots at the edges of their distribution.

Exercise III.10. Show that

$$-\frac{c_{\text{eff}}}{12} = \Theta_k(\mathrm{i}) + \Theta_{\bar{k}}(\mathrm{i})\ ,$$

where the function $\Theta_k(\mathrm{i})$ was defined in Exercise I.5

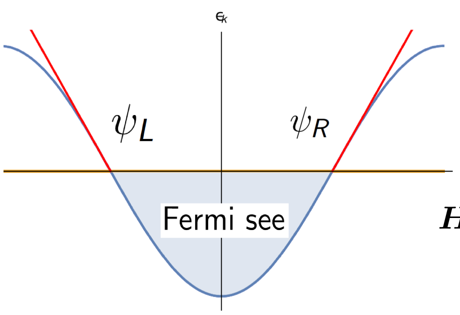
$$k = \tfrac{1}{2}\left(S^z + \tfrac{\theta}{g}\right)\ , \qquad \bar{k} = \tfrac{1}{2}\left(S^z - \tfrac{\theta}{g}\right)\ .$$

Then using the formula (c) from Exercise I.8, one has

$$c_{\text{eff}} = 1 - 12g\left(k^2 + \bar{k}^2\right)\ .$$

Summary of the ODE/IQFT correspondence

Scaling limit: $N,\ J \propto a^{-1} \to \infty,\ \ R = Na$ – fixed



$$\Psi_m \propto \mathrm{e}^{-\mathrm{i}p_R x}\ \psi_R + \mathrm{e}^{-\mathrm{i}p_L x}\ \psi_L^\dagger$$

$$\boldsymbol{H}_{XXZ} \to \boldsymbol{H}_{\text{Thirring}} = \int_0^R \mathrm{d}x\ \Big(\mathrm{i}\,\psi_L^\dagger \partial_x \psi_L - \mathrm{i}\,\psi_R^\dagger \partial_x \psi_R - 2\mathrm{g}_{4\mathrm{F}}\ \psi_L^\dagger \psi_R^\dagger \psi_R \psi_L\Big)$$

The full set of the stationary states in the Thirring IQFT $\leftrightarrow$
a set of pairs of the "monster" potentials $\big(U_{\text{eff}}(z), \bar{U}_{\text{eff}}(\bar{z})\big)$!

References

[1] A. Luther and I. Peschel, "Calculation of critical exponents in two dimensions from quantum field theory in one dimension", Phys. Rev. **B2**, 9, 3908 (1975)

[2] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory