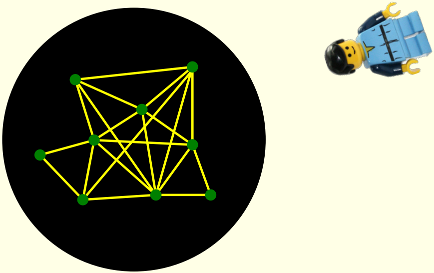
Black holes and the Sachdev-Ye-Kitaev model



Alexei Kitaev (Caltech)

The ultimate goal: a quantum theory of gravity

- This is an important fundamental question: quantum gravity is necessary to desribe processes at very small distances and high energies, of the order of

$$\ell_{\mathrm{P}} \approx 1.62 \cdot 10^{-35}\,\mathrm{m}, \qquad E_{\mathrm{P}} \approx 1.96 \cdot 10^{9}\,\mathrm{J}, \approx 1.22 \cdot 10^{19}\,\mathrm{GeV},$$

  and to understand the early Universe.

- Gravity cannot be quantized by analogy with other fields. The theory must involve some nonlocality at the Planck scale.

- String theory may be the way to go, but it is complex. The non-perturbative string theory is not fully defined yet; there are open problems. [Disclaimer: I am not a string theorist.]

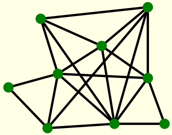
- Can we say something qualitative about quantum gravity before we have a "theory of everything"?

The SYK model

antisymmetric

N Majorana operators χ_j

$$\boxed{\chi_j\chi_k + \chi_k\chi_j = \delta_{jk}}$$



$$H = -\frac{1}{4!}\sum_{j,k,l,m} J_{jklm}\,\chi_j\,\chi_k\,\chi_l\,\chi_m$$

$$\overline{J_{jklm}} = 0, \qquad \overline{J_{jklm}^2} = 3!\,\frac{J^2}{N^3}$$

- Sachdev, Ye, 1993 – a similar model with $\mathrm{SU}(M)$ spins and two-body interactions. (The spins are made fermions.)
- This model (Kitaev, 2015):
  - The same Green function $G(\tau) = -\langle \mathbf{T}\,\chi_j(\tau)\chi_j(0)\rangle$.
  - Disorder effects (replica-off-diagonal terms) are negligible, which simplifies the study of four-point correlators.
  - **Emergent conformal symmetry (for $\beta J \gg 1$) and the corresponding pseudo-Goldstone mode**.
- Detailed calculations: Maldacena, Stanford, `arxiv:1604.07818`,
  Kitaev, Suh, `arXiv:1711.08467`

Why this model?

- Sachdev and Ye were interested in strongly correlated systems at low temperatures.

- My goal was to capture some features of quantum gravity.
  - **Observation:** The semiclassical theory of gravity has some compelling results (e.g. the Hawking radiation) but also some paradoxes.
  - **Concrete goal:** find some fully quantum toy model that would be similar to gravity at the semiclassical level.
  - **Important step:** identify some universal behaviors in semiclassical gravity. One of them is concerned with out-of-time-order (OTO) correlators.

- This $0+1$-dimensional model has a collective mode that is similar to dilaton gravity in $1+1$ dimensions.

Black holes: introduction

- Schwarzschild metric (for $r > a$):

$$\boxed{d\ell^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2}$$

in $3+1$ dimensions,

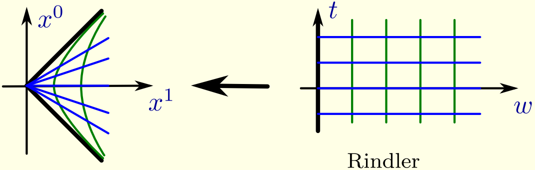
$$f(r) = 1 - \frac{a}{r}, \quad a = 2GM$$

- The apparent singularity at $r = a$ can be removed by a coordinate change.

  – Rindler patch of the Minkowski space:

$$x^0 = \sqrt{\tfrac{2w}{\varkappa}}\,\sinh(\varkappa t)$$

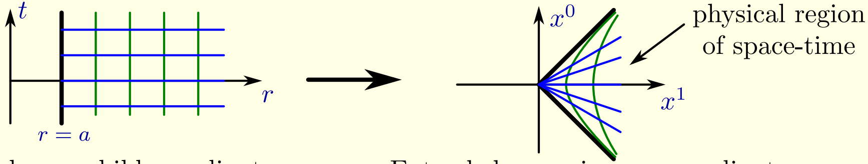
$$x^1 = \sqrt{\tfrac{2w}{\varkappa}}\,\cosh(\varkappa t)$$



$$d\ell^2 = \overbrace{-(dx^0)^2 + (dx^1)^2}^{\text{Minkowski}} = \overbrace{-(2\varkappa w)\,dt^2 + \frac{dw^2}{2\varkappa w}}^{\text{Rindler}}$$

"Eternal" black hole

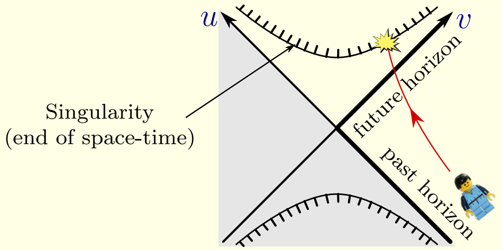
- Near $r = a$, the space is similar to the flat Minkowski space and can be *extended beyond the horizon*.



Schwarzschild coordinates — Extended space in new coordinates

- Time translation acts as a Lorentz boost near the horizon

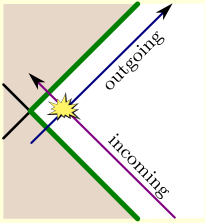
$$\boxed{\text{Surface gravity: } \varkappa = (\text{Lorentz boost at } r = a)/(\text{time at } r = \infty)}$$



Quantum black holes: three levels of understanding

- Quantum fields in classical space (Hawking): $T = \dfrac{\varkappa}{2\pi}, \quad S = \dfrac{A}{4}$

- Gravitational interaction between incoming and outgoing radiation:
  - Dray-t'Hooft shock waves
  - Out-of-time-order correlators (OTOCs): $\langle W(t)\, Y(0)\, Z(t)\, X(0)\rangle$ for $t \lesssim (2\pi T)^{-1} \ln S$
  - Correlations in the Hawking radiation relative to a purifying system, which is a part of the thermofield double state $|\Psi\rangle = Z^{-1/2} \sum_n e^{-E_n/(2T)} |n, n\rangle$



- S-matrix and correlations in the Hawking radiation itself (which appear after half of the black hole has evaporated).

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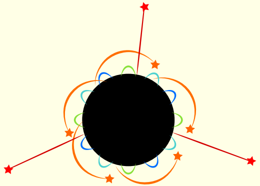
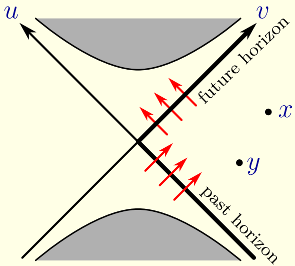
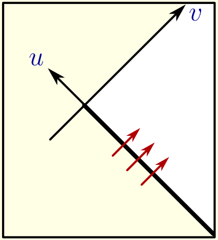
Hawking radiation





- The black hole horizon is a special type of heat bath. It can be characterized by time-dependent correlation functions.
  - Causal correlators of free bosons or fermions, such as $\langle[\psi_\alpha(x), \psi_\beta(y)]\rangle$, are found by solving the wave equation in the physical region.
  - More general correlators, e.g. $\langle\psi_\alpha(x)\psi_\beta(y)\rangle$, depend on the quantum state on the past horizon.
- Assumption: the field correlators at the past horizon are like in the flat space.
- The vacuum (zero-temperature) correlators in the (u, v) coordinates are equivalent to thermal correlators in terms of the global time t.

Hawking radiation (cont.)



- Example: free (Majorana) fermion in $1+1$ D

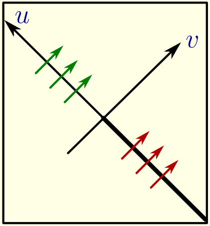
$$\langle \psi_\rightarrow(u,0)\,\psi_\rightarrow(u',0)\rangle \sim (u-u')^{-1},$$

$\boxed{u = -e^{-\varkappa t}}$ on the past horizon,

$$\langle \psi_\rightarrow(t)\,\psi_\rightarrow(t')\rangle \sim \left(\sinh\frac{\pi(t-t')}{\beta}\right)^{-1}, \qquad \boxed{\beta = \frac{2\pi}{\varkappa}}$$

inverse temperature

- Entanglement pattern (perhaps needs revision in the full quantum theory):



  – Radiation entering the physical region is entangled with some field modes behind the horizon.

Black hole thermonynamics

- Thermodynamic quantities (such as temperature and entropy) are related to geometric parameters:
  - horizon area: $A = 4\pi a^2$, where $a = 2GM$;
  - surface gravity: $\varkappa = 1/(2a)$.

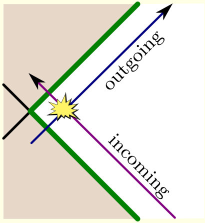
$$\boxed{E = M, \quad T = \frac{\varkappa}{2\pi}, \quad S = \frac{A}{4G}}$$

- Derivation sketch:
  - An identity in classical gravity, $dM = \dfrac{1}{8\pi G}\varkappa\, dA$ looks like the first law of thermodynamics, $dE = T\, dS$.
  - Hawking's relation (quantum): $T^{-1} = \beta = \dfrac{2\pi}{\varkappa}$.

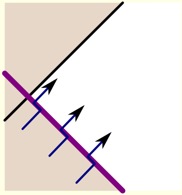
How does information get out of the black hole?

- Gravitational interaction between incoming and outgoing particles is amplified by the Lorentz factor, which grows exponentially with time:

$$\boxed{\gamma = e^{\varkappa t}}$$



- Classically, an initial gravitational perturbation evolves into a “shock wave” at the past horizon, Drey and t’Hooft (1985), t’Hooft (1986).



- Such shock waves don’t alter the quantum state on the past horizon, which is translationally invariant. They show up in *out-of-time-order* (OTO) correlators, e.g. $\langle D(t)\,C(0)\,B(t)\,A(0)\rangle$ (Shenker and Stanford, 2013, 2015).

Naturally ordered (Keldysh) vs. OTO correlators

- Keldysh correlators can be measured by interaction with a probe

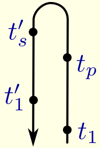
$$H = H_{\text{system}} + H_{\text{probe}} - \sum_j \underset{\uparrow\,\text{system}}{X_j}\, \underset{\nwarrow\,\text{probe}}{Y_j},$$

The probability of some measurement outcome,

$$P = \left\langle \mathbf{T}\exp\left(i\int H(t)dt\right)\,\Pi\,\mathbf{T}\exp\left(-i\int H(t)dt\right)\right\rangle$$

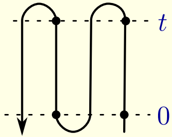
expands into terms like this:

$$\boxed{\begin{gathered}\left\langle X_{j_1}(t'_1)\cdots X_{j_s}(t'_s)\,X_{k_p}(t_p)\cdots X_{k_1}(t_1)\right\rangle_{\text{system}}\\ t'_1<\cdots<t'_s,\quad t_p>\cdots>t_1\end{gathered}}$$



- OTO correlators can only be measured by reversing the time evolution (or using the thermofield double)

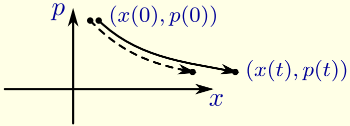
For example, $\boxed{\langle D(t)C(0)B(t)A(0)\rangle}$



Out-of-time-order (OTO) correlators

- First discussed by Larkin and Ovchinnikov (1969). Classically, they describe the divergence of phase space trajectories (a.k.a. the "butterfly effect".)

$$[p_j(t), p_k(0)] = i\hbar\,\frac{\partial p_j(t)}{\partial x_k(0)} \sim \hbar\, e^{\varkappa t}$$



- For typical non-integrable systems with all-to-all interactions:
  - At early times (but after the two-point correlators have decayed):

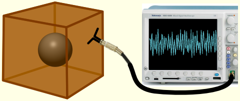
$$\langle D(t)C(0)B(t)A(0)\rangle - \langle DB\rangle\langle CA\rangle \sim \boxed{\frac{1}{N}\, e^{\varkappa t}}$$

    (At later times, the exponential growth saturates.)
  - Upper bound on the growth exponent (Shenker, Stanford, and Maldacena, 2015):

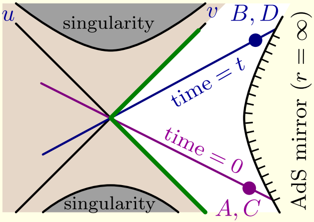
$$\boxed{\varkappa \leqslant 2\pi T}$$

T'Hoofts effect and universality

- The OTO correlators related to t'Hooft's effect are calculated in a well-defined setting: black hole in a "box" (actually, the anti-de Sitter space).



- One considers correlators like $\langle D(t)C(0)B(t)A(0)\rangle$, where the operators A, B, C, D act near the space boundary.

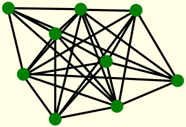


- The growth exponent $\varkappa$ equals the surface gravity, hence $\boxed{\varkappa = 2\pi T}$ due to Hawking's relation.
- Most many-body systems give (much) smaller values, but the SYK model matches the black hole result.

Some failed attempts to fake a black hole

- Random Heisenberg model with all-to all coupling

$$H = -\sum_{j<k}\sum_{\alpha} J_{jk} S_j^\alpha S_k^\alpha, \qquad \overline{J_{jk}^2} = J^2/N.$$



- One can argue that OTO spin correlators go like $N^{-1}e^{\varkappa t}$. But:
  - If $T \gg J$, then $\varkappa \sim J \ll T$.
  - If $T \ll J$, then the system freezes into a spin glass.

- The random Hubbard model becomes a Fermi-liquid at low temperatures (if the coupling is weak) or a spin glass (if the coupling is strong).

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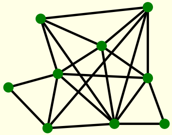

$$H = -\frac{1}{4!}\sum_{j,k,l,m} J_{jklm}\,\chi_j\,\chi_k\,\chi_l\,\chi_m$$

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- Detailed calculations: Maldacena, Stanford, `arxiv:1604.07818`, Kitaev, Suh, `arXiv:1711.08467`

What is special about the SYK model?

- Strongly correlated at low temperatures ($\beta J \gg 1$); not a Fermi liquid or glass
- Exactly solvable:
  - Dynamic mean field approximation for large N
  - Complete analytic solution for $N \gg \beta J \gg 1$

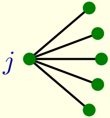
quantum fluctuations are small

Analytic solution of the DMF equations

- Emergent symmetries for $\beta J \gg 1$:
  - The equilibrium Green function $G(\tau_1, \tau_2)$ is $\mathrm{PSL}(2, \mathbb{R})$-invariant. (Here, $\tau \in [0, \beta]$ is the Matsubara time.)
  - Effective mean-field action $I[G]$ is invariant under time reparametrizations (symmetry group $\mathrm{Diff}(S^1)$).
- The $\mathrm{Diff}(S^1) / \mathrm{PSL}(2, \mathbb{R})$ pseudo-Goldstone mode plays the same role as the t'Hooft-Drey shock waves.

Dynamic mean field

- Local field acting on the j-th Majorana mode:



$$\xi_j = -i\,\frac{\partial H}{\partial \chi_j} = -\frac{i}{3!}\sum_{k,l,m} J_{jklm}\chi_k\chi_l\chi_m$$

This is a sum of many small terms, therefore the fluctuating variables $\xi_j(\tau)$ are Gaussian.

- Let

$$G(\tau_1,\tau_2) = -\langle \mathbf{T}\,\chi_j(\tau_1)\chi_j(\tau_2)\rangle, \quad \Sigma(\tau_1,\tau_2) = -\langle \mathbf{T}\,\xi_j(\tau_1)\xi_j(\tau_2)\rangle$$

  Self-consistency (Schwinger-Dyson) equations:

$$\Sigma(\tau_1,\tau_2) = J^2 G(\tau_1,\tau_2)^3, \qquad \hat{G}^{-1} = \underbrace{\hat{G}_0^{-1}}_{\substack{\text{may be neglected} \\ \text{if } \beta J \gg 1}} - \hat{\Sigma}$$

Replica-diagonal effective action for $N \gg 1$

- Dynamic variables: Σ and G.

$$\beta F = -\overline{\ln Z} = -\lim_{M\to 0} \frac{\ln \overline{Z^M}}{M} \quad (M \text{ is the number of replicas})$$

negligible
if $\beta J \gg 1$

$$\boxed{\begin{aligned}\frac{\beta F}{N} = \;& -\ln \mathrm{Pf}(-\partial_\tau - \Sigma) \\ & + \frac{1}{2}\iint \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{4}G(\tau_1,\tau_2)^4\right) d\tau_1\, d\tau_2\end{aligned}}$$

- Stationary points are solutions of the Schwinger-Dyson equations.

A soft (pseudo-Goldstone) mode

- In the zero-temperature limit, $\beta J \to \infty$, the Schwinger-Dyson equations and the effective action are invariant under

$$G(\tau_1,\tau_2) \;\longrightarrow\; G(f(\tau_1),f(\tau_2))\, f'(\tau_1)^{\Delta} f'(\tau_2)^{\Delta}$$

$$\Sigma(\tau_1,\tau_2) \;\longrightarrow\; \Sigma(f(\tau_1),f(\tau_2))\, f'(\tau_1)^{1-\Delta} f'(\tau_2)^{1-\Delta}$$

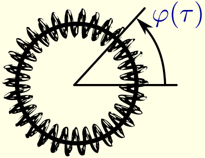
$$\Delta = \frac{1}{4}$$

- The solutions have the form

$$\boxed{G(\tau_1,\tau_2) = G_\infty\big(f(\tau_1),f(\tau_2)\big)\, f'(\tau_1)^{\Delta} f'(\tau_2)^{\Delta}}$$

where $\quad G_\infty(\tau_1,\tau_2) = b(\tau_1-\tau_2)^{-2\Delta}, \qquad f(\tau) = e^{i\varphi(\tau)}.$

- When βJ is large but finite, the energy depends on the map $\varphi:\, S^1 \to S^1$ ("deformation of a spring"). The minimum energy corresponds to $\varphi(\tau) = \dfrac{2\pi}{\beta}\tau.$



Effective action for the pseudo-Goldstone mode

- The effective action is

$$\boxed{I = \beta F = -\alpha_S N J^{-1} \int_0^\beta \mathrm{Sch}\big(e^{i\varphi(\tau)}, \tau\big)\, d\tau}$$

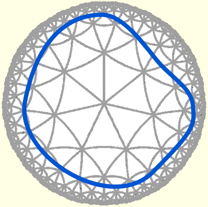
where $\alpha_S \sim 1$ and $\mathrm{Sch}(f(\tau), \tau)$ is the Schwarzian derivative:

$$\mathrm{Sch}(f(\tau), \tau) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

- This effective action also describes a thermally fluctuating string or balloon in the hyperbolic plane:

$$\boxed{\begin{array}{c} I = \alpha_S N\big(\underbrace{L}_{\text{length}} - \underbrace{A}_{\text{area}} - 2\pi\big) \\ \text{constraint:} \quad L = \beta J \end{array}}$$

The minimum is achieved on circles

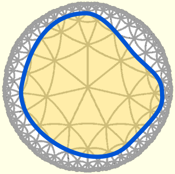


The hyperbolic plane is represented by the Poincare disk model:

$$d\ell^2 = \frac{4(dr^2 + r^2 d\varphi^2)}{(1 - r^2)^2}$$

Connection to gravity in 2 or $1+1$ dimensions

Maldacena, Stanford, Yang (2016)

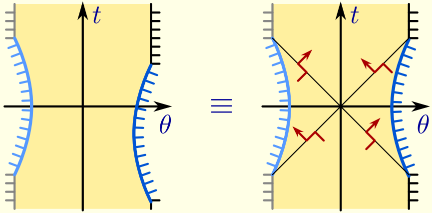


- The effective boundary action can be written using a dilaton field Φ in the bulk:

$$I = -\frac{1}{4\pi}\int_{\text{bulk}} \Phi \underbrace{(R+2)}_{=0} \sqrt{g}\, dx - \frac{1}{2\pi}\int_{\text{boundary}} \Phi \underbrace{K}_{\substack{\text{extrinsic}\\ \text{curvarure}}} d\ell$$

- The Lorentzian version describes quantum fluctuations of the space boundary ("anti-de Sitter mirror").

A stationary configugation with mismatched left and right boundaries is equivalent to a pair of t'Hooft's shock waves:



Summary and further plans

- Some properties of black holes and the SYK model are similar. In particular, the OTO correlators grow in time at the highest possible rate, $\varkappa = 2\pi T$.
- The remainder of this lecture:
  - Feynmann diagrams;
  - Emergent $\mathrm{Diff}(S^1)$ and $\mathrm{PSL}(2,\mathbb{R})$ symmetries.
- Next lecture:
  - The replica-diagonal effective action;
  - Ladder diagrams and conformal kernel.